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KNOTS WITH INFINITELY MANY MINIMAL SPANNING SURFACES

BY

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ABSTRACT. We show that if k_1 and k_2 are nonfibered knots, then the composite knot $K = k_1 \# k_2$ has an infinite collection of minimal spanning surfaces, no two of which are isotopic by an isotopy which leaves the knot K fixed. This result is then applied to show that whether or not a knot has a unique minimal spanning surface can depend on what definition of spanning surface equivalence is used.

Introduction. If K is a polygonal representative of a tame knot in S^3, then K is spanned by a polyhedral, orientable surface [10, §7], [23]; an orientable spanning surface of smallest possible genus is called a *minimal spanning surface* of K, and $g(K)$, the *genus* of K, is then defined as the genus of such a surface. Spanning surfaces F and F' of a knot K are *strongly equivalent* if there is an isotopic deformation of S^3 moving F to F' and leaving K setwise fixed at each level; such an isotopic deformation is then a *strong equivalence*. (We would obtain an equivalent definition of "strongly equivalent" if we had required our strong equivalences to leave K pointwise fixed (compare [22], [28]), for an isotopic deformation which moves F to F' and leaves K setwise fixed can easily be modified to give an isotopic deformation which moves F to F' and leaves K pointwise fixed.) Spanning surfaces F and F' of a knot K are *weakly equivalent* if there is an autohomeomorphism of S^3 taking F to F' and preserving the orientations of S^3 and K.

H. Schubert and K. Soltsien proved in 1964 [22] that a simple knot (i.e., a knot with no nontrivial companions [10, §7], [21]) can have at most finitely many strong equivalence classes of minimal spanning surfaces; they conjectured, however, that this result does not generalize to all knots. We verify this conjecture by showing that if k_1 and k_2 are nonfibered knots, then the composite knot $k_1 \# k_2$ has an infinite collection of minimal spanning surfaces, no two of which are strongly equivalent.

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More explicitly, we construct, in §1, an infinite collection of minimal spanning surfaces for *any* composite knot $k_1 \# k_2$. This construction is similar to one suggested to the author by W. Haken. The basic idea is illustrated in Figures 1, 2, and 3. A composite knot $K = k_1 \# k_2$ can be represented as in Figure 1 [10, §7], which pictures the case where k_1 is the trefoil knot and k_2 is the figure eight knot. A minimal spanning surface F of K can then be constructed by piecing together surfaces S_1 and S_2 corresponding to minimal spanning surfaces of k_1 and k_2, as indicated schematically in Figure 2. From this picture, we can see how to construct an infinite number of variants of F, by adding copies of T to F; four examples are shown in Figure 3. We number these surfaces according to how many "signed" copies of T are added; in particular, $F = F^0$.

If either k_1 or k_2 is fibered, then all of these spanning surfaces are strongly equivalent, as we shall show in §2. The basic principle involved here is that it is possible to "roll a minimal spanning surface of a fibered knot around that knot," by "rolling the surface through the fibration associated with the knot." Using this principle, we can then construct, for each n, a strong equivalence moving F to F^n when either k_1 or k_2 is fibered. For instance, Figure 4 shows how to get a strong equivalence moving F to F^1 when k_2 is fibered, by rolling S_2 through the fibration (of V') associated with k_2. Figure 5 illustrates how to obtain a strong equivalence moving F to F^1 when k_1 is fibered, by rolling S_1 (in the opposite direction) through the fibration (of $V - K$) associated with k_1 (while leaving K fixed).

On the other hand, if neither k_1 nor k_2 is fibered, then no two of the spanning surfaces F^n are strongly equivalent. After establishing some preliminary results in §§3, 4 and 5, we will prove this in §6. There we observe that if J is a strong equivalence moving F^n to F, then $(J_1|S^3 - K)_*$ is an inner automorphism of $\pi_1(S^3 - K)$. As we show in §4, we may assume that $J_1(T) = T$, and this assumption, coupled with the fact that $\pi_1(S^3 - K)$ is a nontrivial free product with amalgamation of $\pi_1(V - K)$ and $\pi_1(V')$, enables us to show, with the aid of a group theoretic result established in §5, that the inner automorphism $(J_1|S^3 - K)_*$ breaks up into inner automorphisms of $\pi_1(V - K)$ and $\pi_1(V')$. In fact, we can arrange that both of these inner automorphisms be given by conjugation by μ^r, where μ is a meridian in $\pi_1(T)$. Now, as we show in §3, the existence of certain kinds of inner automorphisms of a knot group (namely, the kind of inner automorphism which would arise from rolling an orientable spanning surface around the knot) would imply that the commutator subgroup of the knot group was finitely generated, so that, by [24], the knot itself would be fibered. Since J_1 takes F^n to F, we can apply this fact about inner automorphims of knot groups to our inner automorphisms of $\pi_1(V - K)$ and $\pi_1(V')$ to show first that $r = 0$, since k_2 is nonfibered, and second that $n = -r$, since k_1 is nonfibered. Thus, when neither k_1 nor k_2 is fibered, we see that F^n cannot be strongly equivalent to F unless $n = 0$; it is then easy to see that no two of the spanning surfaces F^n are strongly equivalent.

We can also use our result about inner automorphisms of knot groups to show, in §3, that it is impossible to roll a minimal spanning surface of a nonfibered knot around that knot. Together with the results in §2, this provides us with a new characterization of fibered knots: a knot is fibered if and only if it has a minimal spanning surface which can be rolled around the knot.

Finally, in §7, we give two applications. First, if k_1 and k_2 are suitably chosen nonfibered knots (most twist knots will do), then a minimal spanning surface of $k_1 \# k_2$ has a higher Euler characteristic than any other (possibly nonorientable, but still polyhedral) spanning surface of $k_1 \# k_2$; i.e., any minimal spanning surface of $k_1 \# k_2$ is also, in the language of [26], a "spanning surface of maximal characteristic." Thus, for such knots k_1 and k_2, the knot $k_1 \# k_2$ has infinitely many strong equivalence classes of spanning surfaces of maximal characteristic, contradicting the assertion in [26].

In our second application, we show that whether or not a knot has a unique minimal spanning surface can depend on which definition of spanning surface equivalence is used. Namely, if k_1 and k_2 are suitably chosen nonfibered knots (the examples given by W. Whitten in [28] will do), then all minimal spanning surfaces of $k_1 \# k_2$ will be weakly equivalent, even though $k_1 \# k_2$ actually has an *infinite* number of strong equivalence classes of minimal spanning surfaces. Previous examples of nonuniqueness all involved knots which had more than one weak equivalence class of minimal spanning surfaces [2], [3], [6], [7], [13], [27], while all former uniqueness results showed that certain knots had only one strong equivalence class of minimal spanning surfaces (see [5], where it is shown that all fibered knots have this property, [14] and [28]).

The results in this paper are contained in the author's doctoral thesis, which was written at Princeton University under the supervision of the late Professor R. H. Fox, while the author held an N.S.F. graduate fellowship.

I should like to express here my sincerest gratitude to Professor Fox. I would also like to thank Professor H. F. Trotter and Professor W. Whitten.

1. **Construction of the spanning surfaces.** Let k_1 and k_2 be oriented knots in oriented S^3 with minimal spanning surfaces $\tilde{S}_1$ and $\tilde{S}_2$. Let $\tilde{N}$ be a regular neighborhood of a meridian $\tilde{m}$ of k_1, meeting $\tilde{S}_1$ in a disk and not meeting k_1. Then $\tilde{V} = \mathrm{cl}(S^3 - \tilde{N})$ is an unknotted solid torus, with longitude $\tilde{l} = \tilde{S}_1 \cap \partial\tilde{V}$. Let V be a regular neighborhood of k_2, meeting $\tilde{S}_2$ is an annulus, so that $l = \tilde{S}_2 \cap \partial V$ is a longitude of V. Let f: $\tilde{V} \to V$ be a faithful, orientation preserving, P.L. homeomorphism with $f(\tilde{l}) = l$, and set $S_1 = f(\tilde{S}_1 \cap \tilde{V})$, V'

$= \mathrm{cl}(S^3 - V)$, and $S_2 = \tilde{S}_2 \cap V'$. Then $K = f(k_1)$ is the composite knot $k_1 \# k_2$ [10, §7], and, since $g(K) = g(k_1) + g(k_2)$ [10, §7], $F = S_1 \cup S_2$ is a minimal spanning surface of K.

Choose a meridian m on $T = \partial V$ and write T as $m \times l = S^1 \times S^1$, with the orientation of m being consistent with the orientations of K and S^3. Whenever we have a torus given explicitly as $S^1 \times S^1$, the *meridional roll* R of the torus will be the isotopic deformation of the torus given by

$$R_t(\theta_1, \theta_2) = (\theta_1 + 2\pi t, \theta_2);$$

notice that $R_1 = \mathrm{id}$. Since T is collared in V, we may extend R to a P.L. isotopic deformation E of V such that, for each t, $E_t(K) = K$. Extend E_1 to a homeomorphism e of S^3 with $e|V' = \mathrm{id}$, and, for each $n \in \mathbf{Z}$, set $F^n = e^n(F) = (E_1)^n(S_1) \cup S_2$ and $S_1^n = (E_1)^n(S_1)$; each F^n is a minimal spanning surface of K which is weakly equivalent to $F = F^0$, since e^n preserves the orientations of both S^3 and K.

2. **Rolling a surface through a fibration.** Let S be a surface in a connected 3-manifold M with $H_1(M, M - S) = \mathbf{Z}$, and let L be an isotopic deformation of M such that $L_1(S) = S$. Then $w(L, S)$, the winding number of L with respect to S, is defined to be the element of $H_1(M, M - S) = \mathbf{Z}$ represented by the path of a point $x \notin S$ during the isotopy L (which is independent of x).

PROPOSITION 2.1. *Let S be a compact, connected, oriented surface with $\partial S = \cup l_i$, properly embedded in a compact, connected, oriented 3-manifold M whose boundary is the union of tori $T_i = m_i \times l_i = S^1 \times S^1$, where the orientations of the m_i are consistent with the orientation of S and M. If M fibers over S^1 with fiber S, then there is an isotopic deformation L: $M \times I \to M$ such that $L_1(S) = S$ and, for each i, $L|T_i \times I = R_i$, the meridional roll of T_i. Also, for each $n \in \mathbf{Z}$, there is an isotopic deformation L^n: $M \times I \to M$ such that $(L^n)_1(S) = S$ and $w(L^n, S) = n$.*

PROOF. Cut M along S to obtain $\mathfrak{M}$ containing surfaces $\mathfrak{S}$ and $\mathfrak{S}'$ which are copies of S and annuli $\mathfrak{T}_i = \mathrm{m}_i \times l_i = I \times S^1$, with $\partial\mathfrak{S} = 0 \times (\cup l_i)$, and let q: $\mathfrak{M} \to M$ be the natural map. Since M fibers over S^1 with fiber S, there is a homeomorphism r: $I \times S \to \mathfrak{M}$ such that $r(0 \times S) = \mathfrak{S}$, $r(1 \times S) = \mathfrak{S}'$, and $q \circ r_0 = \mathrm{id}_S$. Since $r|0 \times l_i = \mathrm{id}$, $r|I \times l_i$ is isotopic to the identity homeomorphism of the annulus $I \times l_i$ by an isotopy which leaves $0 \times l_i$ pointwise fixed; consequently, since $\cup \mathfrak{T}_i$ is collared in $\mathfrak{M}$, we can arrange that $r|I \times (\cup l_i) = \mathrm{id}$.

Now let $\mathfrak{L}$: $(I \times S) \times I \to (I \times S)$ be the function given by

(1) $\mathfrak{L}(u, s, t) = (u + t, s)$ if $(u + t) < 1$,

(2) $\mathfrak{L}(u, s, t) = (u + t - 1, q \circ r_1(s))$ if $1 \leqslant (u + t)$.

Then $(q \circ r) \circ \mathfrak{L} \circ ((q \circ r)^{-1} \times \mathrm{id})$ is a well-defined, continuous map $L\colon M \times I \to M$ which is in fact an isotopic deformation of M with the required properties.

Finally, for each $n \in \mathbf{Z}$, we define the isotopic deformation L^n by setting $(L^n)_t = (L_t)^n$. $\square$

Thus it is possible to "roll the surface S through a fibration of M."

COROLLARY 2.2. *Let S be a compact, connected, oriented surface with $\partial S = k \cup (\bigcup l_i)$, embedded in a compact, connected, oriented 3-manifold M such that $S \cap \partial M = \bigcup l_i$ and ∂M consists of a union of tori $T_i = m_i \times l_i = S^1 \times S^1$, where the orientations of the m_i are consistent with the orientations of S and M. Let $N(k)$ be a regular neighborhood of k, intersecting S in an annulus. If $\mathrm{cl}(M - N(k))$ is fibered over S^1 with fiber $\mathrm{cl}(S - N(k))$, then there is an isotopic deformation L of M such that $L_1(S) = S$, $L_t(k) = k$ for each t, and, for each i, $L|T_i \times I = R_i$, the meridional roll of T_i. Also, for each $n \in \mathbf{Z}$, there is an isotopic deformation L^n of M such that $(L^n)_1(S) = S$, $(L^n)_t(k) = k$ for each t, and $w(L^n|(M - k) \times I, S - k) = n$.*

PROOF. Write $\partial N(k) = T(k)$ as $m(k) \times l(k) = S^1 \times S^1$, where the orientation of $m(k)$ is consistent with the orientations of S and M, and $l(k) = S \cap T(k)$. Then apply Proposition 2.1 to $\mathrm{cl}(S - N(k)) \subset \mathrm{cl}(M - N(k))$ to obtain an isotopic deformation of $\mathrm{cl}(M - N(k))$ which can then be extended to an isotopic deformation L of M with the required properties, and define L^n by setting $(L^n)_t = (L_t)^n$. $\square$

If S is an orientable spanning surface of a knot k in S^3, then a *strong self-equivalence* of S with winding number n is an isotopic deformation L of S^3 such that $L_1(S) = S$, $L_t(k) = k$ for each t, and $w(L|(S^3 - k) \times I, S - k) = n$. By Corollary 2.2, we have

THEOREM 2.3. *If S is a minimal spanning surface of a fibered knot in S^3, then, for each $n \in \mathbf{Z}$, there is a strong self-equivalence of S with winding number n.* $\square$

Now we will apply these ideas to show that if either k_1 or k_2 is a fibered knot, then all of the surfaces $\bar{F}^n$ are strongly equivalent. First we must establish a preliminary lemma which says that when k_1 is fibered we may assume that the fibration is nice with respect to $\tilde{V}$, so that we may apply Corollary 2.2 to $S_1 \subset V$. For this purpose, let $N(k_1) \subset \tilde{V}$ be a regular neighborhood of k_1, meeting $\tilde{S}_1$ in an annulus, and set $N(K) = f(N(k_1))$. We then have

LEMMA 2.4. *If k_1 is a fibered knot in S^3, then $\mathrm{cl}(V - N(K))$ fibers over S^1 with fiber $\mathrm{cl}(S_1 - N(K))$.*

PROOF. If k_1 is a fibered knot, then $\mathrm{cl}(S^3 - N(k_1))$ fibers over S^1 with fiber $\mathrm{cl}(\tilde{S}_1 - N(k_1))$. Since $\tilde{m}$ is a meridian of k_1, meeting $\tilde{S}_1$ in a point, we may choose the fibration so that $\tilde{m}$ is a section. If $\overline{N}$ is a nice regular neighborhood of $\tilde{m}$, meeting $\tilde{S}_1$ in a disk, then $\mathrm{cl}(S^3 - \overline{N} - N(k_1))$ fibers over S^1 with fiber $\mathrm{cl}(\tilde{S}_1 - \overline{N} - N(k_1))$. Since $\tilde{N}$ is also a regular neighborhood of $\tilde{m}$, meeting $\tilde{S}_1$ in a disk, there is a homeomorphism of S^3 which takes $\overline{N}$ to $\tilde{N}$ and leaves $\tilde{S}_1$ setwise fixed; consequently,

$$\mathrm{cl}(\tilde{V} - N(k_1)) = \mathrm{cl}(S^3 - \tilde{N} - N(k_1))$$

fibers over S^1 with fiber

$$\mathrm{cl}((\tilde{S}_1 \cap \tilde{V}) - N(k_1)) = \mathrm{cl}(\tilde{S}_1 - \tilde{N} - N(k_1)),$$

or, equivalently, $\mathrm{cl}(V - N(K))$ fibers over S^1 with fiber $\mathrm{cl}(S_1 - N(K))$. $\square$

THEOREM 2.5. *If either k_1 or k_2 is a fibered knot, then, for each $n \in \mathbf{Z}$, F^n is strongly equivalent to F.*

PROOF. If k_1 is a fibered knot, $\mathrm{cl}(V - N(K))$ fibers over S^1 with fiber $\mathrm{cl}(S_1 - N(K))$. By Corollary 2.2, there is then an isotopic deformation L of V such that $L_1(S_1) = S_1$, $L_t(K) = K$ for each t, and $L|T \times I = R = E|T \times I$. For each $n \in \mathbf{Z}$, let J^n be the isotopic deformation of S^3 given by

(1) $(J^n)_t|V = (E_t)^n \circ (L_t)^{-n}$,

(2) $(J^n)_t|V' = \mathrm{id}_{V'}$.

Since $L|T \times I = E|T \times I$, J^n is well defined. Also, $(J^n)_t(K) = K$ for each t, and $(J^n)_1(F) = (J^n)_1(S_1 \cup S_2) = (E_1)^n \circ (L_1)^{-n}(S_1) \cup S_2 = (E_1)^n(S_1) \cup S_2 = F^n$, so that, for each $n \in \mathbf{Z}$, F^n is strongly equivalent to F.

If k_2 is a fibered knot, then V' fibers over S^1 with fiber S_2. By Proposition 2.1, there is an isotopic deformation L of V' such that $L_1(S_2) = S_2$ and $L|T \times I = R = E|T \times I$. For each $n \in \mathbf{Z}$, let J^n be the isotopic deformation of S^3 given by

(1) $(J^n)_t|V = (E_t)^n$,

(2) $(J^n)_t|V' = (L_t)^n$.

Since $L|T \times I = E|T \times I$, J^n is well defined. Also, $(J^n)_t(K) = K$ for each t, and $(J^n)_1(F) = (J^n)_1(S_1 \cup S_2) = (E_1)^n(S_1) \cup (L_1)^n(S_2) = (E_1)^n(S_1) \cup S_2 = F^n$, so that, for each $n \in \mathbf{Z}$, F^n is strongly equivalent to F. $\square$

3. An obstruction to rolling.

PROPOSITION 3.1. *Let k be a knot in S^3, let S be an orientable spanning surface of k, and let $N(k)$ be a regular neighborhood of k, intersecting S in an annulus. Let M be either $(S^3 - k)$ or $\mathrm{cl}(S^3 - N(k))$, set $Y = M - S$, take $y \in Y$, and set $U = i_*(\pi_1(Y,y)) \subset \pi_1(M,y)$, where i: $Y \to M$ is the inclusion map. Suppose $\zeta \in \pi_1(M,y)$ can be represented by a simple closed curve z which pierces S exactly once, and, for some $n \neq 0$, $\zeta^{-n} U \zeta^n = U$. Then k is a fibered knot.*

PROOF. Let $\hat{M}$ be the infinite cyclic covering space of M. Let $\hat{y}_i$ and $\hat{Y}_i$ ($i \in \mathbf{Z}$) denote the various lifts of y and Y, indexed so that the lift $\hat{z}^i$ of z^i which begins at $\hat{y}_0$ ends at $\hat{y}_i \in \hat{Y}_i$. Set $\hat{Y}'_i = \mathrm{cl}(\hat{Y}_i) \cup \hat{z}^i$.

Applying the Seifert-Van Kampen theorem, we see that $\pi_1(\hat{M},\hat{y}_0)$ is generated by $\pi_1(\hat{Y}'_i,\hat{y}_0)$ ($i \in \mathbf{Z}$). Now

$$\pi_1(\hat{Y}'_i,\hat{y}_0) = \hat{z}^i \cdot (\pi_1(\hat{Y}'_i,\hat{y}_i)) \cdot \hat{z}^{-i} = \hat{z}^i \cdot (\pi_1(\hat{Y}_i,\hat{y}_i)) \cdot \hat{z}^{-i}.$$

Projecting down to $\pi_1(M,y)$, we see that $\pi_1(\hat{Y}_i,\hat{y}_i)$ goes to U, and hence $\pi_1(\hat{Y}'_i,\hat{y}_0)$ goes to $\zeta^i U \zeta^{-i}$. Hence $\pi_1(\hat{M},\hat{y}_0)$ (viewed as a subgroup of $\pi_1(M,y)$) is generated by $\zeta^i U \zeta^{-i}$ ($i \in \mathbf{Z}$).

However, $\zeta^{-n} U \zeta^n = U$, where $n \neq 0$, so $\pi_1(\hat{M},\hat{y}_0)$ is generated by

$$U, \zeta U \zeta^{-1}, \ldots, \zeta^{(|n|-1)} U \zeta^{-(|n|-1)}.$$

Since U is finitely generated, we conclude that $[\pi_1(M,y),\pi_1(M,y)] = \pi_1(\hat{M},\hat{y}_0)$ is finitely generated, so that, by [24], k is a fibered knot. □

THEOREM 3.2. *Let k be a knot in S^3 with orientable spanning surface S, and suppose that for some $n \neq 0$, there is a strong self-equivalence L of S with winding number n. Then k is a fibered knot.*

PROOF. Let $N(k)$ be a regular neighborhood of k intersecting S in an annulus. Take a point $y \in (S^3 - S)$ close enough to k so that $L(y \times I) \subset N(k)$. $\pi_1(N(k) - k,y)$ is free on two generators ζ and λ, where ζ can be represented by a simple closed curve z which pierces S exactly once and λ can be represented by a simple closed curve which is parallel to k and does not meet S. By following L by an isotopy which leaves S fixed, we see that we can assume that $L_1(y) = y$ and also that $L(y \times I)$ represents ζ^r for some r. Since $w(L|(S^3 - k) \times I, S - k) = n$, we must have $r = n$.

Since L restricts to an isotopic deformation of $(S^3 - k)$ and ζ^n is the element of $\pi_1(S^3 - k,y)$ represented by the path of y during this isotopic deformation, $(L_1|S^3 - k)_*$ is the inner automorphism of $\pi_1(S^3 - k,y)$ given by $\eta \to \zeta^{-n}\eta\zeta^n$. Since $L_1(S) = S$, $L_1(S^3 - S) = (S^3 - S)$, so, letting i: $(S^3 - S) \to (S^3 - k)$ be the inclusion map, we have

$$(L_1|S^3 - k)_*(i_*(\pi_1(S^3 - S,y))) = i_*(\pi_1(S^3 - S,y)),$$

or

$$\zeta^{-n}(i_*(\pi_1(S^3 - S,y)))\zeta^n = i_*(\pi_1(S^3 - S,y)).$$

By Proposition 3.1, k must be a fibered knot. □

Combining Theorems 2.3 and 3.2 yields a new characterization of fibered knots: a knot is fibered if and only if it has a minimal spanning surface admitting a strong self-equivalence with nonzero winding number.

4. Getting a torus back where it belongs. Take a point $\tilde{x} \in \partial\tilde{V} - \tilde{l}$ and set $x = f(\tilde{x}) \in T - l$; x and $\tilde{x}$ will serve as basepoints in §§5 and 6. Fix a metric ρ on S^3. Whenever c_1 and c_2 are continuous maps of a compact space S into S^3, we set

$$\rho^*(c_1, c_2) = \sup_{s \in S} \rho(c_1(s), c_2(s)).$$

PROPOSITION 4.1. *Suppose k_2 is not the trivial knot, and suppose J is a strong equivalence moving F^n to F. Then there is an isotopic deformation of S^3 moving $J_1(T)$ to T, moving $J_1(x)$ to x, and leaving F setwise fixed at each level.*

Consequently, if k_2 is not the trivial knot and F^n is strongly equivalent to F, there is a strong equivalence $\mathfrak{J}$ which moves F^n to F and satisfies the additional condition that $\mathfrak{J}_1(T, x) = (T, x)$.

REMARK. In order to isotop $J_1(T)$ to T while leaving F setwise fixed, it is clearly necessary that we be able to isotop $J_1(T) \cap F = J_1(T \cap F^n) = J_1(l)$ to $T \cap F = l$ on F. Thus, before proving Proposition 4.1, we first prove the following

LEMMA 4.2. *Under the hypotheses of Proposition* 4.1, *$J_1(l)$ is freely homotopic to l on F.*

REMARK. Write $S^3 = \mathbf{R}^3 \cup \infty$, set $B_1 = \{(y_1, y_2, y_3): y_1 \geqslant 0\} \cup \infty$, $B_2 = \{(y_1, y_2, y_3): y_1 \leqslant 0\} \cup \infty$, $S^2 = \{(y_1, y_2, y_3): y_1 = 0\} \cup \infty$, $a = \{(y_1, y_2, y_3): y_1 = 0, -1 \leqslant y_2 \leqslant 1, y_3 = 0\}$, and let h be the autohomeomorphism of S^3 given by $h(y_1, y_2, y_3) = (-y_1, -y_2, y_3)$. Now suppose that in our construction in §1 we had k_2 and $\tilde{S}_2$ lying in B_2, with $k_2 \cap S^2 = \tilde{S}_2 \cap S^2 = a$, and that we took V to be a regular neighborhood of $(B_1 \cup k_2)$. If, furthermore, $k_1 = k_2$ and $\tilde{S}_1 = \tilde{S}_2$, we could easily arrange that $K = f(k_1) = (h(k_2) \cup k_2) - \mathrm{int}(a)$, and that S_1 be the union of $h(\tilde{S}_2)$ and $\tilde{S}_2 \cap V$. Then h takes $F = S_1 \cup S_2 = h(\tilde{S}_2) \cup \tilde{S}_2$ to itself, while preserving the orientations of S^3 and K, but $h(l)$ is not freely homotopic to l on F, provided that k_1 and k_2 are nontrivial. Indeed, l is parallel to k_2 on $\tilde{S}_2$, so l represents a nontrivial element of $\pi_1(\tilde{S}_2)$, and $h(l)$ represents a nontrivial element of $\pi_1(h(\tilde{S}_2))$. Since $F = \tilde{S}_2 \cup h(\tilde{S}_2)$ and $\tilde{S}_2 \cap h(\tilde{S}_2) = a$, $\pi_1(F) = \pi_1(\tilde{S}_2) * \pi_1(h(\tilde{S}_2))$, and hence l and $h(l)$ cannot represent conjugate elements of $\pi_1(F)$, so that l and $h(l)$ cannot be freely homotopic on F.

Then $h \circ e^{-n}$ is an autohomeomorphism of S^3 which takes F^n to F while preserving the orientations of S^3 and K (i.e., $h \circ e^{-n}$ is a weak equivalence taking F^n to F), but $(h \circ e^{-n})(l) = h(e^{-n}(l)) = h(l)$ is not freely homotopic to l on F. Therefore, in proving Lemma 4.2, it will be essential to make use of the fact that J_1 is isotopic to the identity by an isotopy fixing K. Indeed, the main idea of our argument is to trace the intersection curves of T with F as T moves about during the isotopy J.

PROOF OF LEMMA 4.2. Note that $K \cap J(V' \times I) = \Phi$, since, for each t, $K \cap J_t(V') = J_t(K) \cap J_t(V') = J_t(K \cap V') = J_t(\Phi) = \Phi$. Therefore, since $J(V' \times I)$ is compact, we may take a regular neighborhood Q of K, intersecting F in an annulus, such that Q is disjoint from $J(V' \times I)$. Set $Q' = \mathrm{cl}(S^3 - Q)$, set $F' = Q' \cap F$, and let $F' \times [-1,1]$ be a regular neighborhood of F' in Q', with $F' = F' \times 0$ and $T \cap (F' \times [-1,1]) = l \times [-1,1]$. Set $F^+ = F' \times (\frac{1}{2})$ and $F^- = F' \times (-\frac{1}{2})$, and let ε_1 be the minimum of the following positive numbers:

$$\rho(Q' - (F' \times (-\tfrac{1}{2},\tfrac{1}{2})), F'), \quad \rho(Q' - (F' \times (0,1)), F^+),$$
$$\rho(Q' - (F' \times (-1,0)), F^-), \quad \rho(Q' - (F' \times (-1,1)), F' \times [-\tfrac{1}{2},\tfrac{1}{2}]).$$

Let $p_1\colon F' \times [-1,1] \to F'$ be the natural projection map, and let p_2 be a deformation retraction of F' onto an r-leafed rose $G_r \subset S^3$. Triangulate G_r finely enough so that the star of any simplex in G_r is a tree, and let $\varepsilon_3 > 0$ be a small enough number so that the ε_3-neighborhood (in the metric ρ) of any simplex of G_r is contained in the star of that simplex. Now, if c_1 and c_2 are maps of S^1 to $G_r \subset S^3$ with $\rho^*(c_1,c_2) < \varepsilon_3$, then c_1 and c_2 are freely homotopic. Indeed, for any $s \in S^1$, $c_1(s)$ lies in some edge of G_r, and then $c_2(s)$ is in the star of that edge, which is a tree; there is a unique line segment joining $c_1(s)$ to $c_2(s)$ in this tree. If $c_1(s)$ is a vertex of G_r, then, since $c_2(s)$ must be in the star of this vertex, we see that the construction of the line segment joining $c_1(s)$ to $c_2(s)$ does not depend on which edge we take $c_1(s)$ to lie in. Thus, to each $s \in S^1$ we can associate a unique line segment joining $c_1(s)$ to $c_2(s)$. For each $s \in S^1$, push $c_1(s)$ to $c_2(s)$ along such a line segment. This procedure gives a homotopy between c_1 and c_2.

Now $F' \times [-1,1]$ is compact, so $p_2 \circ p_1$ is uniformly continuous. Therefore, there is a positive number ε_2 such that if c_1 and c_2 are maps of S^1 to $F' \times [-1,1]$ with $\rho^*(c_1,c_2) < \varepsilon_2$, then

$$\rho^*(p_2 \circ p_1 \circ c_1, p_2 \circ p_1 \circ c_2) < \varepsilon_3,$$

and consequently $p_2 \circ p_1 \circ c_1$ and $p_2 \circ p_1 \circ c_2$ are freely homotopic. In that case, since p_2 is a deformation retraction, $p_1 \circ c_1$ and $p_1 \circ c_2$ are freely homotopic as maps from S^1 to F.

Let $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$. Since $S^3 \times I$ is compact, $J\colon S^3 \times I \to S^3$ is uniformly continuous, so there is a $\delta > 0$ such that $|t_1 - t_2| < \delta \Rightarrow \rho^*(J_{t_1}, J_{t_2}) < \varepsilon/3$. Take a positive integer j with $(1/j) < \delta$. Using Theorem 5 of [16] and general position arguments we can find, for each integer i, $1 \leqslant i \leqslant j-1$, an autohomeomorphism u_i of S^3 such that $\rho^*(u_i, \mathrm{id}) < \varepsilon/3$, $u_i|Q = \mathrm{id}$, and $u_i(J_{i/j}(T))$ is polyhedral and in general position with respect to F', F^+, and F^-. Set $u_0 = u_j = \mathrm{id}$, set $v_i = u_i \circ J_{i/j}$ for $0 \leqslant i \leqslant j$, and set $h_i = v_{i+1} \circ (v_i)^{-1}$, for $0 \leqslant i \leqslant j-1$. For each i, $0 \leqslant i \leqslant j-1$, we have $\rho^*(h_i, \mathrm{id}) < \varepsilon$, since $\rho^*(h_i, \mathrm{id}) = \rho^*(v_{i+1}, v_i) \leqslant \rho^*(v_{i+1}, J_{(i+1)/j}) + \rho^*(J_{(i+1)/j}, J_{i/j}) + \rho^*(J_{i/j}, v_i) = \rho^*(u_{i+1}, \mathrm{id}) + \rho^*(J_{(i+1)/j}, J_{i/j}) + \rho^*(u_i, \mathrm{id}) \leqslant \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$. For each i, $0 \leqslant i \leqslant j$, we have $Q \subset \mathrm{int}(v_i(V))$, since $u_i|Q = \mathrm{id}$ and $Q \cap J_{i/j}(V') = \Phi$; in particular $v_i(T) \subset \mathrm{int}(Q')$. Also, for each i, $0 \leqslant i \leqslant j$, K has winding number one in $v_i(V)$, since $K = J_{i/j}(K) = v_i(K)$. For each i, $0 \leqslant i \leqslant j-1$, $v_i(T)$ intersects F', F^+, and F^- in a disjoint collection of simple closed curves, since $v_i(T) \subset \mathrm{int}(Q')$ and $v_i(T)$ is in general position with respect to F', F^+, and F^-. Finally, even though $v_j(T)$ might not be polyhedral, $v_j(T)$ intersects F' in exactly one simple closed curve; indeed,

$$v_j(T) \cap F' = v_j(T) \cap F = J_1(T) \cap F = J_1(T \cap F'') = J_1(l).$$

On each $v_i(T)$ we distinguish two types of simple closed curves: α-curves, which bound disks on $v_i(T)$, and β-curves, which do not. Let $\mathfrak{n}_i$ be the number of simple closed curves in $v_i(T) \cap F$ which are freely homotopic to l on F. Note that any such curve must be a β-curve. Indeed, if $\alpha \subset v_i(T) \cap F$ bounds a disk on $v_i(T)$, then, since F is minimal, α must bound a disk on F. Consequently, α cannot be freely homotopic to l on F, or else l would also bound a disk on F and then k_2 would be the trivial knot. Observe as well that any α-curve in $v_i(T) \cap (F^+ \cup F^-)$ also bounds a disk in $F^+ \cup F^-$, again by the minimality of F. Note finally that there must be at least one β-curve in $v_i(T) \cap F^+$ and one β-curve in $v_i(T) \cap F^-$, for otherwise K would be nullhomologous in $v_i(V)$.

We now show that, if $0 \leqslant i \leqslant j-1$, then $\mathfrak{n}_i = \mathfrak{n}_{i+1} \pmod 2$. We start by observing that the β-curves in $v_i(T) \cap (F^+ \cup F^-)$ divide $v_i(T)$ into various annuli, which we shall call *brackets*. A bracket B may intersect $(F^+ \cup F^-)$ in some α-curves, each of which bounds a disk D' in B and a disk D'' in $(F^+ \cup F^-)$. Let B' be the closure of (B minus the disks D'), and let B'' be obtained from B' by recapping the holes with appropriate disks D''. (B'' may be singular; this is all right.) We say that B is an *outer bracket* if $B' \subset Q' - (F' \times (-\frac{1}{2}, \frac{1}{2}))$, and that B is an *inner bracket* if $B' \subset F' \times [-\frac{1}{2}, \frac{1}{2}]$. An inner bracket B will be called an *even inner bracket* if either $\partial B \subset F^+$ or $\partial B \subset F^-$, while B will be an *odd inner bracket* if ∂B consists of one curve in F^+ and one curve in F^-.

Any β-curve in $v_i(T) \cap F = v_i(T) \cap F'$ must be contained in B' for some bracket $B \subset v_i(T)$. For a given bracket $B \subset v_i(T)$, let $\mathfrak{n}(B)$ be the number of β-curves in $B' \cap F'$. For an outer bracket B, $\mathfrak{n}(B) = 0$. If B is an even inner bracket, then $\mathfrak{n}(B)$ is even, while if B is an odd inner bracket, then $\mathfrak{n}(B)$ is odd.

Now $v_{i+1}(T) = h_i(v_i(T))$. Also, for any bracket $B \subset v_i(T)$, $\partial(h_i(B')) \cap F' = \Phi$, since

$$\rho^*(h_i, \mathrm{id}) < \varepsilon \leqslant \varepsilon_1 \leqslant \rho(Q' - (F' \times (-\tfrac{1}{2}, \tfrac{1}{2})), F') \leqslant \rho(F^+ \cup F^-, F').$$

Consequently, any β-curve in $v_{i+1}(T) \cap F = v_{i+1}(T) \cap F'$ lies in $h_i(B')$, for some bracket $B \subset v_i(T)$. For a given bracket $B \subset v_i(T)$, let $\mathfrak{n}'(B)$ be the number of β-curves in $h_i(B') \cap F'$. If B is an outer bracket, then, since $\rho^*(h_i, \mathrm{id}) < \varepsilon \leqslant \varepsilon_1 < \rho(Q' - (F \times (-\frac{1}{2}, \frac{1}{2})), F')$, $\mathfrak{n}'(B) = 0$. If B is an inner bracket, then, since

$$\rho^*(h_i, \mathrm{id}) < \varepsilon \leqslant \varepsilon_1 \leqslant \rho(Q' - (F' \times (-1, 1)), F' \times [-\tfrac{1}{2}, \tfrac{1}{2}]),$$

$h_i(B') \subset F' \times (-1, 1)$. Furthermore, since

$$\begin{aligned} \rho^*(h_i, \mathrm{id}) &< \varepsilon \leqslant \varepsilon_1 \\ &\leqslant \min(\rho(Q' - (F' \times (0,1)), F^+),\ \rho(Q' - (F' \times (-1,0)), F^-)), \end{aligned}$$

we see that if B is an even inner bracket, then either $\partial(h_i(B)) \subset F' \times (0, 1)$ or $\partial(h_i(B)) \subset F' \times (-1, 0)$, while if B is an odd inner bracket, then $\partial(h_i(B))$ consists of one curve in $F' \times (0, 1)$ and one curve in $F' \times (-1, 0)$. Consequently, if B is an even inner bracket, then $\mathfrak{n}'(B)$ is even, while if B is an odd inner bracket, then $\mathfrak{n}'(B)$ is odd.

Now we shall show that if B is an inner bracket in $v_i(T)$, then all of the β-curves in $B' \cap F'$ and all of the β-curves in $h_i(B') \cap F'$ are freely homotopic on F. Indeed, let B be an inner bracket in $v_i(T)$, and fix a β-curve $\beta_0 \subset B'$. If β_1 is any β-curve in B', then β_1 is freely homotopic to β_0 on B, and hence β_1 is freely homotopic to β_0 on $B'' \subset F' \times [-1, 1]$. Projecting this homotopy to F' by p_1 shows that $p_1(\beta_1)$ is freely homotopic to $p_1(\beta_0)$ on F. In particular, if β_1 is a β-curve in $B' \cap F'$, then $\beta_1 = p_1(\beta_1)$ is freely homotopic to $p_1(\beta_0)$ on F. If β_2 is any β-curve in $h_i(B') \subset F' \times [-1, 1]$, then $(h_i)^{-1}(\beta_2)$ is a β-curve in $B' \subset F' \times [-1, 1]$. Since $\rho^*(h_i, \mathrm{id}) < \varepsilon \leqslant \varepsilon_2$, $p_1(\beta_2)$ is freely homotopic to $p_1((h_i)^{-1}(\beta_2))$ on F. Also, since $(h_i)^{-1}(\beta_2)$ is a β-curve in B', $p_1((h_i)^{-1}(\beta_2))$ is freely homotopic to $p_1(\beta_0)$ on F. Thus $p_1(\beta_2)$ is freely homotopic to $p_1(\beta_0)$ on F. In particular, if β_2 is a β-curve in $h_i(B') \cap F'$, then $\beta_2 = p_1(\beta_2)$ is freely homotopic to $p_1(\beta_0)$ on F.

Thus we have: if B is an outer bracket, then $\mathfrak{n}(B) = \mathfrak{n}'(B) = 0$, while if B is an inner bracket, then $\mathfrak{n}(B) = \mathfrak{n}'(B) \pmod 2$ and all of the β-curves in $B' \cap F'$ and all of the β-curves in $h_i(B') \cap F'$ are freely homotopic on F. Therefore $\mathfrak{n}_i = \mathfrak{n}_{i+1} \pmod 2$, as desired.

Now $v_0(T) \cap F' = T \cap F = l$, so $\mathfrak{n}_0 = 1$. Also, as we remarked earlier, $v_j(T) \cap F' = J_1(l)$. Since we must have $\mathfrak{n}_j = 1 \pmod 2$, we see that $J_1(l)$ must be freely homotopic to l on F. $\square$

PROOF OF PROPOSITION 4.1. The construction of the desired isotopic deformation will be done in four steps. First, using Lemma 4.2, we isotop J_1 so that $J_1(l) = l$, simultaneously getting $J_1|F^n$ to be P.L. Then we can isotop J_1 to a P.L. autohomeomorphism of S^3, getting $J_1(T)$ to intersect T in a nice way. At this point it is possible to use classical methods of three dimensional topology to isotop $J_1(T)$ to T. Finally, we get $J_1(x)$ back to x by simply pulling along T.

Step 1. By Theorem A4 of [9], there is an isotopic deformation τ of F such that $\tau_1 \circ (J_1|F^n)$ is P.L. Plainly $\tau_1(J_1(l))$ is freely homotopic to $J_1(l)$ on F, so, by Lemma 4.2, $\tau_1(J_1(l))$ is freely homotopic to l on F. Also, since k_2 is not the trivial knot, l cannot bound a disk on F. Therefore, by Theorem 2.1 of [9], there is a P.L. isotopic deformation ν of F such that $\nu_1 \circ \tau_1(J_1(l)) = l$. Performing τ and ν in succession gives an isotopic deformation of F which extends to an isotopic deformation $\mathfrak{A}$ of S^3 such that $\mathfrak{A}_1 \circ J_1(l) = l$, $\mathfrak{A}_1 \circ J_1|F^n$ is P.L., and, for each t, $\mathfrak{A}_t(F) = F$. Set $\gamma = \mathfrak{A}_1 \circ J_1$.

Step 2. Cut S^3 along F to obtain a manifold M whose boundary consists of two copies of F; similarly, cut S^3 along F^n to obtain a manifold M^n. Let $q\colon M \to S^3$ and $q^n\colon M^n \to S^3$ be the natural maps. Let $\partial M \times [0,1]$ and $\partial M^n \times [0,1]$ be collar neighborhoods of ∂M and ∂M^n, with $\partial M = \partial M \times 0$, $q^{-1}(T) \cap (\partial M \times [0,1]) = q^{-1}(l) \times [0,1]$, $\partial M^n = \partial M^n \times 0$, and $(q^n)^{-1}(T) \cap (\partial M^n \times [0,1]) = (q^n)^{-1}(l) \times [0,1]$.

Since $\gamma(F^n) = F$, γ gives rise to a homeomorphism $\Gamma\colon M^n \to M$. Using the method originated by Alexander in [1], we obtain an isotopy $\Psi\colon M^n \times I \to M$ such that $\Psi_0 = \Gamma$, $\Psi_t|\partial M^n = \Gamma|\partial M^n$ for each t, and Ψ_1 takes $\partial M^n \times [0,3/4]$ to $\partial M \times [0,3/4]$ by sending $w \times s$ in $\partial M^n \times [0,3/4]$ to $\Gamma(w) \times s$ in $\partial M \times [0,3/4]$. Since $\gamma(l) = l$,

$$\Psi_1((q^n)^{-1}(l) \times [0,3/4]) = q^{-1}(l) \times [0,3/4];$$

since $\gamma|F^n$ is P.L., $\Psi_1|\partial M^n \times [0,3/4]$ is P.L. Using a P.L. isotopic deformation of M which fixes everything but $\partial M \times (1/4,3/4)$, we can isotop Ψ_1 to a homeomorphism Λ such that

$$\Lambda((q^n)^{-1}(l) \times (1/4,3/4)) \cap (q^{-1}(l) \times (1/4,3/4)) = \Phi.$$

Given $\varepsilon > 0$, we can use Theorem 4 of [4] and general position arguments to find a P.L. homeomorphism $\Omega\colon M^n \to M$ such that $\Omega|\partial M^n \times [0,1/2] = \Lambda|\partial M^n \times [0,1/2]$, $\rho^*(\Omega,\Lambda) < \varepsilon$, and $\Omega((q^n)^{-1}(T))$ is in general position with respect to $q^{-1}(T)$ in $M - (\partial M \times [0,1/4])$. If ε is chosen sufficiently small, then we may use the second corollary of the main theorem of [12] to obtain an isotopy $\kappa\colon M^n \times I \to M$ with $\kappa_0 = \Lambda$, $\kappa_1 = \Omega$, and $\kappa_t|\partial M^n = \Lambda|\partial M^n = \Gamma|\partial M^n$ for each t. Putting together all our isotopies, we obtain an isotopy

$\Sigma: M^n \times I \to M$, with $\Sigma_0 = \Gamma$, $\Sigma_1 = \Omega$, and $\Sigma_t | \partial M^n = \Gamma | \partial M^n$. Then Σ gives rise to an isotopy $\sigma: S^3 \times I \to S^3$ such that $\sigma_0 = \gamma$, $\sigma_t | F^n = \gamma | F^n$ for each t, and $\sigma_1 = \omega$, the P.L. autohomeomorphism of S^3 which arises from the homeomorphism Ω. Now T and $\omega(T)$ are polyhedral, and we note that $\omega(T)$ is in general position with respect to T off of $q(\partial M \times [0, 1/4])$, while

$$\begin{aligned}\omega(T) \cap q(\partial M \times [0, 1/4]) &= q(q^{-1}(l) \times [0, 1/4]) \\ &= T \cap q(\partial M \times [0, 1/4]),\end{aligned}$$

an annulus A_0, so that $T \cap \omega(T)$ consists of A_0 and a collection of simple closed curves at which T and $\omega(T)$ meet transversely, and $\omega(T) \cap F = T \cap F = l \subset A_0$. Note also that $\omega(l) = l$ and, since $\omega(F^n) = F$, $\omega(K) = K$. Finally, observe that $\mathfrak{B} = \sigma \circ (\gamma^{-1} \times \mathrm{id}): S^3 \times I \to S^3$ is an isotopic deformation of S^3 which leaves F pointwise fixed at each level, with $\omega = \mathfrak{B}_1 \circ \gamma = \mathfrak{B}_1 \circ \mathfrak{A}_1 \circ J_1$.

Steps 3 *and* 4. We first note that a simple closed curve $c \subset T \cap \omega(T)$ bounds a disk on T if and only if it bounds a disk on $\omega(T)$. Indeed, if c bounds a disk on T, then c is unknotted, while if c does not bound a disk on T, then c is parallel to l on T, so c has the same knot type as k_2. Since these same remarks hold for $\omega(T)$, we see that $c \subset T \cap \omega(T)$ bounds a disk on T iff c is unknotted iff c bounds a disk on $\omega(T)$.

Now suppose $c \subset (T \cap \omega(T)) - A_0$ bounds a disk on T; we take c to bound an innermost such disk D_1. Then c also bounds a disk $D_2 \subset \omega(T)$, and $D_1 \cap D_2 = \partial D_1 \cap \partial D_2$, since D_1 is innermost. Since $\omega(T) \cap F = T \cap F = l \subset A_0$ and $D_1 \cap A_0 = D_2 \cap A_0 = \Phi$, $D_1 \cup D_2$ is a 2-sphere meeting neither F nor A_0; consequently, $D_1 \cup D_2$ bounds a 3-cell not meeting $F \cup A_0$. Pushing D_2 across this 3-cell to D_1 (and then slightly past D_1) gives an isotopic deformation of S^3 which leaves F and A_0 pointwise fixed and moves $\omega(T)$ so as to remove the intersection curve c, without creating any new intersections of $\omega(T)$ and T. We see that all intersection curves in $(T \cap \omega(T)) - A_0$ which bound disks on T may be removed in this way.

The remaining curves in $(\omega(T) \cap T) - \mathrm{int}(A_0)$ (with $\omega(T)$ altered by the procedure in the last paragraph) divide $\omega(T) - \mathrm{int}(A_0)$ into annuli, with every other annulus lying in V. Take such an annulus $A \subset V$, with $\partial A = c_1 \cup c_2$; c_1 and c_2 divide T into two annuli A_1 and A_2. Since c_1 and c_2 are in $T - \mathrm{int}(A_0)$, either A_1 or A_2, say A_1, must lie entirely in $T - \mathrm{int}(A_0)$; then we have $A_0 \subset A_2$. Since c_1 and c_2 are both parallel to l on T, they both have winding number one on V. Therefore, by Theorem 1 on p. 207 of [21], $A \cup A_1$ bounds a solid torus $W \subset V$, and the winding number of c_1 and c_2 on W is one.

Now $\partial W \cap (F \cup \mathrm{int}(A_0)) = \Phi$, because, even after our alterations of $\omega(T)$

(which left F and A_0 pointwise fixed), we still have $\omega(T) \cap F = T \cap F = l \subset \text{int}(A_0)$, and $\partial W = A \cup A_1$, where $A \subset \omega(T) - \text{int}(A_0)$ and $A_1 \subset T - \text{int}(A_0)$. Also, l, which is in $F \cup \text{int}(A_0)$, is not in W, since $l \subset \text{int}(A_0) \subset \text{int}(A_2)$. Therefore, $W \cap (F \cup \text{int}(A_0)) = \Phi$, and $W \cap (F \cup A_0) = \partial A_0 \cap \partial A$.

Since the winding number of c_1 and c_2 on W is one, we can push A across the solid torus W to A_1, holding ∂A fixed; we then push $A - (\partial A_0 \cap \partial A)$ slightly past A_1 into $\text{int}(V')$, leaving $(\partial A_0 \cap \partial A)$ fixed. Since $W \cap (F \cup A_0) = \partial A_0 \cap \partial A$, we obtain in this way an isotopic deformation of S^3 which leaves F and A_0 pointwise fixed and moves $\omega(T)$ so as to push $A - (\partial A_0 \cap \partial A)$ into $\text{int}(V')$. In the process, other annuli $A' \subset \omega(T) - \text{int}(A_0)$ may also be pushed into $\text{int}(V')$, but no annuli will be pushed back into V, and no new intersection curves (of $\omega(T)$ with T) will be created. By a succession of such isotopic deformations, we can alter $\omega(T)$ so that $\omega(T) - \text{int}(A_0) \subset \text{int}(V')$, and hence $\omega(T) \cap T = \omega(T) \cap V = A_0$.

Since $V \cap \omega(T) = A_0 \subset \partial V$, either $V \subset \omega(V)$ or $V \subset \omega(V')$. However, $K \subset \text{int}\, V$, and, since none of our alterations moved K, we have $K = \omega(K) \subset \omega(V)$. Therefore, $V \subset \omega(V)$. By Theorem 1 on p. 207 of [21], the annuli $T - \text{int}(A_0)$ and $\omega(T) - \text{int}(A_0)$ cobound a solid torus $W^* \subset \omega(V)$, with $W^* \cup V = \omega(V)$. Furthermore, letting $\partial(T - \text{int}(A_0)) = \partial(\omega(T) - \text{int}(A_0)) = \partial A_0 = c_{01} \cup c_{02}$, we see that the curves c_{01} and c_{02} have winding number one on $\omega(V)$; indeed, they are parallel to l on $\omega(T)$, and, as all of our alterations left F pointwise fixed, we still have $l = \omega(l)$, which has winding number one on $\omega(V)$. Therefore, again using Theorem 1 on p. 207 of [21], we conclude that c_{01} and c_{02} have winding number one on W^*.

Now $\partial W^* \cap (F \cup \text{int}(A_0)) = \Phi$, because we still have $\omega(T) \cap F = T \cap F = l \subset \text{int}(A_0)$, and $\partial W^* = (T - \text{int}(A_0)) \cup (\omega(T) - \text{int}(A_0))$. Also, l, which is in $F \cup \text{int}(A_0)$, is not in W^*, since $l \subset \text{int}(A_0)$. Therefore $W^* \cap (F \cup \text{int}(A_0)) = \Phi$, and consequently $W^* \cap (F \cup A_0) = \partial A_0 = c_{01} \cup c_{02}$.

Since the winding number of c_{01} and c_{02} on W^* is one, we can push $\omega(T) - \text{int}(A_0)$ across W^* to $T - \text{int}(A_0)$, holding $c_{01} \cup c_{02}$ fixed. Since $W^* \cap (F \cup A_0) = c_{01} \cup c_{02}$, we obtain in this way an isotopic deformation of S^3 which leaves F and A_0 fixed and moves $\omega(T)$ so as to push $\omega(T) - \text{int}(A_0)$ to $T - \text{int}(A_0)$, so that $\omega(T) = T$, as desired.

Finally, since $\omega(x) \in \omega(T - F^n) = T - F$ and $x \in T - F$ we may move $\omega(x)$ back to x on T, while leaving T setwise fixed and F pointwise fixed. □

5. Free products with amalgamation.

PROPOSITION 5.1. *If neither k_1 nor k_2 is trivial, then $\pi_1(S^3 - K, x)$ is a* *nontrivial free product with amalgamation*:

$$\pi_1(S^3 - K, x) = \pi_1(V - K, x) *_{\pi_1(T,x)} \pi_1(V', x).$$

PROOF. Applying the Seifert-Van Kampen theorem to $S^3 - K = (V - K) \cup V'$, we see that we just need to show that the natural maps $\pi_1(T, x) \to \pi_1(V - K, x)$ and $\pi_1(T, x) \to \pi_1(V', x)$ are injective and not surjective.

If $\pi_1(T, x) \to \pi_1(V - K, x)$ were not injective, then by Dehn's lemma and the loop theorem [25, p. 131], there would be a disk $\Delta \subset (V - K)$ with $\partial\Delta \subset T$ representing a nontrivial element of $\pi_1(T)$; since $\partial\Delta$ would be nullhomologous in V, $\partial\Delta$ would have to be a meridian of T, so that K would be nullhomologous in V, which is a contradiction. Hence $\pi_1(T, x) \to \pi_1(V - K, x)$ is injective.

If $\pi_1(T, x) \to \pi_1(V', x)$ were not injective, then there would be a disk $\Delta' \subset V'$ with $\partial\Delta' \subset T$ representing a nontrivial element of $\pi_1(T)$; since $\partial\Delta'$ would be nullhomologous in V', $\partial D'$ would have to be a longitude of T, so that k_2 would be trivial. Hence $\pi_1(T, x) \to \pi_1(V', x)$ is injective.

Applying the Seifert-Van Kampen theorem to $(S^3 - k_1) = (\tilde{V} - k_1) \cup \tilde{N}$, we see that the natural map $\pi_1(\tilde{V} - k_1, \tilde{x}) \to \pi_1(S^3 - k_1, \tilde{x})$ is surjective. If $\pi_1(T, x) \to \pi_1(V - K, x)$ were surjective, then

$$\pi_1(T, x) \to \pi_1(V - K, x) \overset{\cong}{\to} \pi_1(\tilde{V} - k_1, \tilde{x}) \to \pi_1(S^3 - k_1, \tilde{x})$$

would be surjective, so, since $\pi_1(T, x)$ is abelian, $\pi_1(S^3 - k_1, \tilde{x})$ would be abelian, and k_1 would be trivial ([8, Theorem 2, p. 158] and [18]). Hence $\pi_1(T, x) \to \pi_1(V - K, x)$ is not surjective.

If $\pi_1(T, x) \to \pi_1(V', x)$ were surjective, then $\pi_1(S^3 - k_2, x) \cong \pi_1(V', x)$ would be abelian, and k_2 would be trivial. Hence $\pi_1(T, x) \to \pi_1(V', x)$ is not surjective. □

PROPOSITION 5.2. *Let $G = A *_C B$ be a nontrivial free product with amalgamation. If $g \in G$ normalizes both A and B, then $g \in C$.*

PROOF. If $g \notin C$, then, by the normal form theorem ([15, Chapter 4, §2] or [19, Chapter 11, §4]), g can be written as an alternating product of elements a_i in $A - C$ and elements b_i in $B - C$. By symmetry, we may assume that g starts with $a_1 \in A - C$, so either $g = a_1 \cdots b_n$ $(n \geqslant 1)$ or $g = a_1 \cdots a_n$ $(n \geqslant 1)$. Take $b \in B - C = B - A$. Then $b' = g^{-1}b^{-1}g \in B - C$, since g normalizes both A and B. Also $g^{-1}bgb' = 1$. If $g = a_1 \cdots a_n$, then we have $(a_n)^{-1} \cdots (a_1)^{-1} ba_1 \cdots a_n b' = 1$, which is impossible, by the normal form theorem. If $g = a_1 \cdots b_n$, then we have $(b_n)^{-1} \cdots (a_1)^{-1} ba_1 \cdots b_n b' = 1$, so, by the normal form theorem, $b_n b' \in C$, and hence $(a_n)' = a_n b_n b' \in A - C$. Then $(b_n)^{-1} \cdots (a_1)^{-1} ba_1 \cdots (a_n)' = 1$, which is impossible (even if $n = 1$), by the normal form theorem. Thus we must have $g \in C$. □

6. Pairwise nonequivalence.

LEMMA 6.1. *If F^p is strongly equivalent to F^q, then F^{p-q} is strongly equivalent to $F = F^0$.*

PROOF. If J is a strong equivalence moving F^p to F^q, then $e^{-q} \circ J \circ (e^q \times \mathrm{id})$ is a strong equivalence moving F^{p-q} to F. □

THEOREM 6.2. *If k_1 and k_2 are nonfibered knots, then F^p is strongly equivalent to F^q only if $p = q$. Therefore, if k_1 and k_2 are nonfibered knots, then the composite knot $k_1 \# k_2$ has an infinite collection of minimal spanning surfaces, no two of which are strongly equivalent.*

PROOF. Set $n = p - q$. If F^p is strongly equivalent to F^q, then F^n is strongly equivalent to F, so, by Proposition 4.1, there is a strong equivalence J moving F^n to F and satisfying the additional condition that $J_1(T,x) = (T,x)$. Since $J_1(K) = K$ and $J_1(T) = T$, $J_1(V-K) = (V-K)$ and $J_1(V') = V'$.

Since $(J_1|S^3 - K)$ is isotopic to the identity autohomeomorphism of $(S^3 - K)$, $(J_1|S^3 - K)_*$ is the inner automorphism of

$$\pi_1(S^3 - K, x) = \pi_1(V - K, x) *_{\pi_1(T,x)} \pi_1(V', x)$$

given by $\eta \to \xi^{-1}\eta\xi$, where ξ is the element of $\pi_1(S^3 - K, x)$ represented by the path of x during the isotopy J. Since $J_1(V-K) = (V-K)$ and $J_1(V') = V'$,

$$(J_1|S^3 - K)_*(\pi_1(V - K, x)) = \pi_1(V - K, x)$$

and

$$(J_1|S^3 - K)_*(\pi_1(V', x)) = \pi_1(V', x),$$

or $\xi^{-1}(\pi_1(V-K,x))\xi = \pi_1(V-K,x)$ and $\xi^{-1}(\pi_1(V',x))\xi = \pi_1(V',x)$; consequently, by Propositions 5.1 and 5.2, $\xi \in \pi_1(T,x)$. By following J by an isotopy which leaves F pointwise fixed and leaves T setwise fixed, we can arrange that $\xi = \mu^r$, where μ is a meridian in $\pi_1(T,x)$.

Now $(J_1|V')_*$ is the inner automorphism of $\pi_1(V',x)$ given by $\eta \to \mu^{-r}\eta\mu^r$. Since $J_1(F^n) = F$ and $J_1(V') = V'$, $J_1(V' - S_2) = (V' - S_2)$, so, letting i_2: $(V' - S_2) \to V'$ be the inclusion map, we have

$$(J_1|V')_*((i_2)_*(\pi_1(V' - S_2, x))) = (i_2)_*(\pi_1(V' - S_2, x)),$$

or

$$\mu^{-r}((i_2)_*(\pi_1(V' - S_2, x)))\mu^r = (i_2)_*(\pi_1(V' - S_2, x)).$$

By Proposition 3.1, $r = 0$, or else k_2 would be a fibered knot. Hence $(J_1|S^3 - K)_* = \mathrm{id}$.

In particular, $(J_1|V-K)_*$ is the identity automorphism of $\pi_1(V-K,x)$. Since $J_1(F^n)=F$ and $J_1(V-K)=(V-K)$, $J_1(V-S_1^n)=(V-S_1)$, so, letting (i_1^n): $(V-S_1^n)\to(V-K)$ and i_1: $(V-S_1)\to(V-K)$ be the inclusion maps, we have

$$(J_1|V-K)_*((i_1^n)_*(\pi_1(V-S_1^n,x)))=(i_1)_*(\pi_1(V-S_1,x)),$$

or

$$(i_1^n)_*(\pi_1(V-S_1^n,x))=(i_1)_*(\pi_1(V-S_1,x)).$$

Let E^n be the isotopic deformation of V given by $(E^n)_t=(E_t)^n$. Note that E^n leaves K setwise fixed at each level, and moves S_1 to S_1^n. Since $((E^n)_1|V-K)$ is isotopic to the identity autohomeomorphism of $(V-K)$ and μ^n is the element of $\pi_1(V-K,x)$ which is represented by the path of x during the isotopy E^n, $((E^n)_1|V-K)_*$ is the inner automorphism of $\pi_1(V-K,x)$ given by $\eta\to\mu^{-n}\eta\mu^n$. Since $(E^n)_1(S_1)=S_1^n$, $(E^n)_1(V-S_1)=(V-S_1)$, so

$$((E^n)_1|V-K)_*((i_1)_*(\pi_1(V-S_1,x)))=(i_1^n)_*(\pi_1(V-S_1^n,x)),$$

or

$$\mu^{-n}((i_1)_*(\pi_1(V-S_1,x)))\mu^n=(i_1^n)_*(\pi_1(V-S_1^n,x)).$$

However, we also have that

$$(i_1^n)_*(\pi_1(V-S_1^n,x))=(i_1)_*(\pi_1(V-S_1,x)),$$

so we see that

$$\mu^{-n}((i_1)_*(\pi_1(V-S_1,x)))\mu^n=(i_1)_*(\pi_1(V-S_1,x)).$$

Equivalently, letting $\tilde{\imath}$: $(\tilde{V}-\tilde{S}_1)\to(\tilde{V}-k_1)$ be the inclusion map and letting $\tilde{\mu}$ be a meridian in $\pi_1(\partial\tilde{V},\tilde{x})$, we have

$$\tilde{\mu}^{-n}(\tilde{\imath}_*(\pi_1(\tilde{V}-\tilde{S}_1,\tilde{x})))\tilde{\mu}^n=\tilde{\imath}_*(\pi_1(\tilde{V}-\tilde{S}_1,\tilde{x})).$$

Finally, let i: $(S^3-\tilde{S}_1)\to(S^3-k_1)$, o: $(\tilde{V}-\tilde{S}_1)\to(S^3-\tilde{S}_1)$, and u: $(\tilde{V}-k_1)\to(S^3-k_1)$ be the inclusion maps. Applying the Seifert-Van Kampen theorem to $(S^3-\tilde{S}_1)=(\tilde{V}-\tilde{S}_1)\cup(\tilde{N}-\tilde{S}_1)$, we see that o_*: $\pi_1(\tilde{V}-\tilde{S}_1,\tilde{x})\to\pi_1(S^3-\tilde{S}_1,\tilde{x})$ is surjective, so that

$$i_*(\pi_1(S^3-\tilde{S}_1,\tilde{x}))=i_*(o_*(\pi_1(\tilde{V}-\tilde{S}_1,\tilde{X})))=u_*(\tilde{\imath}_*(\pi_1(\tilde{V}-\tilde{S}_1,\tilde{x}))).$$

Hence

$$\begin{aligned}\tilde{\mu}^{-n}(i_*(\pi_1(S^3 - \tilde{S}_1, \tilde{x})))\tilde{\mu}^{-n} &= \tilde{\mu}^{-n}(u_*(\tilde{\imath}_*(\pi_1(\tilde{V} - \tilde{S}_1, \tilde{x}))))\tilde{\mu}^{n} \\ &= u_*(\tilde{\mu}^{-n}(\tilde{\imath}_*(\pi_1(\tilde{V} - \tilde{S}_1, \tilde{x})))\tilde{\mu}^{n}) \\ &= u_*(\tilde{\imath}_*(\pi_1(\tilde{V} - \tilde{S}_1, \tilde{x}))) \\ &= i_*(\pi_1(S^3 - \tilde{S}_1, \tilde{x})).\end{aligned}$$

By Proposition 3.1, $n = 0$, or else k_1 would be a fibered knot; therefore $p = q$. □

COROLLARY 6.3. *There are knots which have infinitely many strong equivalence classes of minimal spanning surfaces.*

PROOF. We only need an example of a nonfibered knot. See [28]. □

7. **Applications.** A (not necessarily orientable, but still polyhedral) spanning surface of a knot of highest possible Euler characteristic is called a *maximal characteristic spanning surface* of k. Then $\chi(k)$ is defined to be the Euler characteristic of such a surface.

Let S^3 be the union of two P.L. 3-balls B_1 and B_2 which intersect in a 2-sphere S^2 containing an arc a. Let k_1 and k_2 be knots in S^3, with $k_1 \subset B_1$, $k_2 \subset B_2$, and $k_1 \cap S^2 = k_2 \cap S^2 = a$, and let K be the composite knot $k_1 \# k_2 = (k_1 \cup k_2) - \mathrm{int}(a)$. (Note: In this section, K, F, S_1, and S_2 will no longer refer to the specific objects constructed in §1.) For both of our applications, we will need the following lemma, which is a generalization of Theorem 4 of [20].

LEMMA 7.1. *Suppose F is either* (1) *a maximal characteristic spanning surface of K or* (2) *a minimal spanning surface of K. Then F is strongly equivalent to a spanning surface of the form $S_1 \cup S_2$, where $S_1 \subset B_1$, $S_2 \subset B_2$, and $S_1 \cap S^2 = S_2 \cap S^2 = a$. In case* (1), *$S_1$ is a maximal characteristic spanning surface of k_1 and S_2 is a maximal characteristic spanning surface of k_2, while in case* (2), *S_1 is a minimal spanning surface of k_1 and S_2 is a minimal spanning surface of k_2. In particular, $\chi(K) = \chi(k_1) + \chi(k_2) - 1$, and $g(K) = g(k_1) + g(k_2)$.*

PROOF. Without moving K, we can arrange that F be in general position with respect to S^2, so that $F \cap S^2$ consists of an arc a' and a collection of simple closed curves. By isotoping a' to a in S^2 while leaving $\partial a' = S^2 \cap K = \partial a$ fixed, we obtain an isotopic deformation of S^3 which leaves K pointwise fixed at each level and moves F so that $F \cap S^2$ consists of the arc a and a collection of simple closed curves in $S^2 - a$.

Each simple closed curve c in $F \cap S^2$ bounds a disk in $(S^2 - a) \subset (S^3 - K)$; take c to bound an innermost such disk D. The curve c must separate F into two components, for otherwise, cutting F along c and gluing on two copies of D, we obtain a spanning surface F' for K, with F' being orientable if F is, such that $\chi(F') = \chi(F) + 2$, which is impossible in either case (1) or case (2). Furthermore, the component which does not contain K must be a disk D', for otherwise, replacing $D' \subset F$ with the disk D, we would obtain a spanning surface F'' for K, with F'' being orientable if F is, such that $\chi(F'') > \chi(F)$, which is impossible. Since D is innermost, $D \cap D' = \partial D$, and hence $D \cup D'$ is a 2-sphere. Since $(D \cup D') \cap (K \cup a) = \Phi$, $D \cup D'$ bounds a 3-cell meeting neither K nor a; pushing D' across this 3-cell to D (and then slightly past D) gives an isotopic deformation of S^3 which leaves K and a pointwise fixed at each level and moves F so as to remove the intersection curve c, without creating any new intersections of F and S^2. Continuing in this manner, we can remove all the simple closed curves from the intersection of F and S^2.

Thus, by isotopic deformations which left K pointwise fixed at each level, we have moved F so that it intersects S^2 in exactly the arc a. Therefore, our original surface F is strongly equivalent to a spanning surface of the form $S_1 \cup S_2$, where $S_1 \subset B_1$ is a spanning surface for k_1 which intersects S^2 in the arc a and $S_2 \subset B_2$ is a spanning surface for k_2 which intersects S^2 in the arc a.

In case (1), let S_1' be a maximal characteristic spanning surface of k_1 and let S_2' be a maximal characteristic spanning surface of k_2, with $S_1' \subset B_1$, $S_2' \subset B_2$, and $S_1' \cap S^2 = S_2' \cap S^2 = a$. If $\chi(S_1') > \chi(S_1)$, or $\chi(S_2') > \chi(S_2)$, then $\chi(S_1' \cup S_2') > \chi(S_1 \cup S_2) = \chi(F)$, a contradiction. Therefore, in case (1), S_1 and S_2 are maximal characteristic spanning surfaces. In particular, we have $\chi(K) = \chi(F) = \chi(S_1) + \chi(S_2) - 1 = \chi(k_1) + \chi(k_2) - 1$.

Similarly, in case (2) we see that S_1 and S_2 must be minimal spanning surfaces, and hence $g(K) = g(F) = g(S_1) + g(S_2) = g(k_1) + g(k_2)$. □

LEMMA 7.2. *If k is a nontrivial twist knot (i.e., a nontrivial double of the trivial knot* [10, §7]) *other than the trefoil knot* 3_1 *or the figure eight knot* 4_1, *then k is a genus one, nonfibered knot, with $\chi(k) = -1$.*

PROOF. Using the procedure in [23], we find an orientable spanning surface F of k, with $\chi(F) = -1$, or $g(F) = 1$. Since k is nontrivial, k has genus one. Furthermore, $\chi(k) = -1$, unless k bounds a moebius band. However, if k bounds a moebius band M, then, thickening M slightly, we see that k must be either a torus knot or a cable knot. By Theorem 4 on p. 580 of [23], the genus of a (p, q) torus knot is $(p-1)(q-1)/2$, so the only torus knot of genus one is the trefoil knot; hence k cannot be a torus knot. Furthermore, k is not a cable knot, for, by Theorem 4 on p. 242 of [21], k is simple. Therefore, k cannot bound a moebius band, and consequently $\chi(k) = -1$.

Finally, since the only fibered knots of genus one are 3_1 and 4_1 [11, Theorem, p. 76], k is a nonfibered knot. □

THEOREM 7.3. *If k_1 and k_2 are nontrivial twist knots other than the trefoil knot or the figure eight knot, then the composite knot $K = k_1 \# k_2$ has an infinite* *collection of maximal characteristic spanning surfaces, no two of which are strongly equivalent.*

PROOF. Since $g(K) = g(k_1) + g(k_2) = 2$ while $\chi(K) = \chi(k_1) + \chi(k_2) - 1 = -3$, any minimal spanning surface of K is also a maximal characteristic spanning surface of K. Since neither k_1 nor k_2 is fibered, we apply Theorem 6.2 to finish the proof. $\square$

Next, we shall show that whether or not a knot has a unique minimal spanning surface can depend on which definition of spanning surface equivalence is used. Let us say that a knot k is strongly unispannable if any two minimal spanning surfaces of k are strongly equivalent. Similarly, k is weakly unispannable if any two minimal spanning surfaces of k are weakly equivalent.

THEOREM 7.4. *There are knots which are weakly unispannable but not strongly unispannable.*

PROOF. W. Whitten has provided examples of nonfibered knots which are strongly unispannable [28]. If k_1 and k_2 are such knots, then the composite knot $K = k_1 \# k_2$ is not strongly unispannable, by Theorem 6.2; indeed, since neither k_1 nor k_2 is fibered, K actually has infinitely many strong equivalence classes of minimal spanning surfaces. However, K is weakly unispannable. Indeed, suppose F and F' are any minimal spanning surfaces of K. By Lemma 7.1, F is strongly equivalent to a spanning surface of the form $S_1 \cup S_2$, where S_1 is a minimal spanning surface of k_1, S_2 is a minimal spanning surface of k_2, $S_1 \subset B_1$, $S_2 \subset B_2$, and $S_1 \cap S^2 = S_2 \cap S^2 = a$. Similarly, F' is strongly equivalent to a spanning surface of the form $S_1' \cup S_2'$, where S_1' is a minimal spanning surface of k_1, S_2' is a minimal spanning surface of k_2, $S_1' \subset B_1$, $S_2' \subset B_2$, and $S_1' \cap S^2 = S_2' \cap S^2 = a$. Now k_1, being strongly unispannable, is certainly weakly unispannable, so S_1 is weakly equivalent to S_1'. Using the argument employed by Schubert in the proofs of Lemmas 5 and 6 on pp. 65–69 of [20], we see that we can in fact take our weak equivalence to be the identity on B_2; i.e., there is an autohomeomorphism h_1 of S^3 which takes S_1 to S_1', preserves the orientations of S^3 and k_1, and is the identity on B_2. Similarly, there is an autohomeomorphism h_2 of S^3 which takes S_2 to S_2', preserves the orientations of S^3 and k_2, and is the identity on B_1. Then $h_2 \circ h_1$ takes $S_1 \cup S_2$ to $S_1' \cup S_2'$ and preserves the orientations of S^3 and K, and, consequently, F and F' are weakly equivalent. $\square$

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